Theory of Equations

1. Solve $x^4 - 5x^3 + 4x^2 + 8x - 8 = 0$, having given that one of the roots is $1+\sqrt{5}$.

2. Solve $z^4 - 4z^2 + 8z - 4 = 0$, given that $1 + i$ is a root.

3. Let $f(x) \equiv ax^3 + bx^2 + cx + d$ and $f(0) = -2$, $f(1) = -6$, $f(-2) = -12$, solve the equation $f(x) = 0$, given that the product of its roots is twice their sum.

4. Solve $x^3 - 5x^2 - 16x + 80 = 0$ if the sum of two roots is zero.

5. Show that the equation $x^4 - 12x^2 + 12x - 3 = 0$ has a root between -3 and -4 and a root between 2 and 3.

6. If the equation $2x^2 - qx + r = 0$ has roots $a + 1$, $b + 2$, where a, b are the real roots of the equation $x^2 - mx + n = 0$ and $a \geq b$, find q, r in terms of m, n.
In the case $a = b$, show that $q^2 = 4(2r + 1)$.

7. By a suitable substitution, express the equation $\frac{x^2}{x+1} + \frac{2x+2}{x^2} - 3 = 0$ in quadratic form and hence solve it completely.

8. If the equations $ax^2 + by^2 = 1$, $Lx + My = 1$ have only one distinct solution for x and y, prove that $\frac{L^2}{a} + \frac{M^2}{b} = 1$, and find the solution.

9. Prove that $\frac{5}{2x^2 + 3x + 3}$ is positive for all real values of x and find its greatest value. Draw a rough graph of the function for values x between -6 and 3.

10. If α, β are the roots of $x^2 + 7x - 3 = 0$, prove that $\alpha^3 + \beta^3 + 7(\alpha^2 + \beta^2) - 3(\alpha + \beta) = 0$.

11. If the roots of $ax^2 + bx + c = 0$ are in the ratio $p : q$, prove that $ac(p + q)^2 = b^2 pq$.

12. Solve $3(x - x^{-1}) = x^2 + x^{-2}$ by the substitution $y = x - x^{-1}$.

13. If the roots of $ax^2 + bx + c = 0$ are such that their difference is one-half of the sum of their reciprocals. Prove that $b^2 (4c^2 - a^2) = 16ac^3$.

14. The roots of $px^2 + x + q = 0$ are α, β. Without solving the equation, find, in terms of p and q, the value of $(p\alpha^3 + q\alpha) + (p\beta^3 + q\beta)$.

15. Prove that, if all roots of the simultaneous equations : $x^2 + y^2 = 1$, $Lx + My = 1$ are real, then $L^2 + M^2 \geq 1$.

16. **(a)** If $f(x) = \dfrac{x^2 + 2ax + b}{x^2 + 1}$, where a and b are real and $a \neq 0$, show that there are two real numbers k such that $f(x) - k$ is of the form $\dfrac{(Ax + B)^2}{x^2 + 1}$.

(b) If these numbers are k_1 and k_2, where $k_1 < k_2$, show that $f(x) - k_r = \dfrac{[(1-k_r)x + a]^2}{(1-k_r)(x^2+1)}$ for $r = 1, 2$. Prove also that $(1 - k_1)(1 - k_2) = - a^2$ and deduce that for real values of x, $k_1 \leq f(x) \leq k_2$.

(c) Make a rough sketch of the curve $y = f(x)$ for $a > 0$, indicating the position of the curve with respect to the lines $y = 1$, $y = k_1$, $y = k_2$.

17. If a and b are unequal real numbers whose sum is not zero, show that

$$(a - b)x^2 - 2(a^2 + b^2)x + (a^3 - b^3) = 0$$

has real or complex roots according as a and b have the same or opposite signs.

Show that the difference between the roots is $\dfrac{2(a+b)\sqrt{ab}}{a-b}$.

18. The roots of $ax^2 + bx + c = 0$ are α and β . Express in terms of α and β the roots of

$$x + 2 + \frac{1}{x} = \frac{b^2}{ac} .$$

19. Find the ranges of values of θ in the interval $0 \leq \theta \leq 2\pi$ for which the equation in x

$x^2 \cos^2\theta + ax\left(\sqrt{3}\cos\theta + \sin\theta\right) + a^2 = 0$ are real.

20. Prove that if α and β are the roots of the equation $ax^2 + bx + c = 0$ and $\alpha : \beta = \lambda : \mu$, then $\lambda\mu\, b^2 = (\lambda + \mu)^2\, ca$.

Deduce the condition (in terms of a, b, c, a', b', c') for the roots of the above equation to be in the same ratio as those of $a'x^2 + b'x + c' = 0$

21. If a is a constant such that $0 < a < 1$, prove that the roots of the equation $(1 - a)x^2 + x + a = 0$ are always real and negative.

Find the equation whose roots are the squares of the reciprocals of the roots of this equation.

22. If α, β, γ are roots of $x^3 + px + q = 0$, form the equation whose roots are $\dfrac{\beta}{\gamma} + \dfrac{\gamma}{\beta}$, $\dfrac{\alpha}{\gamma} + \dfrac{\gamma}{\alpha}$, $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$.

(Hint : $y + 2 = \dfrac{\beta^2 + \gamma^2 + 2\beta\gamma}{\beta\gamma} = \dfrac{\alpha^2}{\beta\gamma} = -\dfrac{\alpha^3}{q} = \dfrac{p\alpha}{q} + 1$.)

23. If α and β are roots of $ax^2 + bx + c = 0$, find the value of :

(a) $\dfrac{1}{(a\alpha + b)^2} + \dfrac{1}{(a\beta + b)^2}$ **(b)** $\dfrac{1}{(a\alpha + b)^3} + \dfrac{1}{(a\beta + b)^3}$.

24. If α and β are roots of $3x^2 + x - 1 = 0$, prove that $3(\alpha^3 + \beta^3) + (\alpha^2 + \beta^2) - (\alpha + \beta) = 0$, and $3(\alpha^4 + \beta^4) + (\alpha^3 + \beta^3) - (\alpha^2 + \beta^2) = 0$. Hence find the values of $\alpha^3 + \beta^3$ and $\alpha^4 + \beta^4$.

25. Given that a, b, c are the roots of the equation $x^3 + px^2 + qx + r = 0$, express $a^3 + b^3 + c^3$ in terms of p, q, r and show that $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = \frac{3pqr - q^3 - 3r^2}{r^3}$.

26. Prove that the equation whose roots are the four numbers obtained by adding a root of $x^2 + 2ax + b = 0$ to a root of $x^2 + 2px + q = 0$ is $[x^2 + 2(a + p)x + b + q + 2ap]^2 - 4(a^2 - b)(p^2 - q) = 0$.

27. If a, b, c are the roots of the equation $x^3 + mx + n = 0$, prove that

$$a^6 + b^6 + c^6 = \frac{1}{3}\left(a^3 + b^3 + c^3\right)^2 + \frac{1}{2}\left(a^2 + b^2 + c^2\right)\left(a^4 + b^4 + c^4\right) .$$

28. If $1, x_1, x_2, x_3, x_4$ are the roots of the equation $x^5 - 1 = 0$, prove that

$(1 - x_1)(1 - x_2)(1 - x_3)(1 - x_4) = 5$ and $(1 + x_1)(1 + x_2)(1 + x_3)(1 + x_4) = 1$.

29. If the roots $x^n = 1$ are $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$, prove that $(1 - \alpha_1)(1 - \alpha_2) \ldots (1 - \alpha_{n-1}) = n$.

30. If $a_1, a_2, \ldots, a_n$ are the roots of $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0$, show that

$(1 + a_1^2)(1 + a_2^2) \ldots (1 + a_n^2) = (1 - p_2 + p_4 - \ldots)^2 + (p_1 - p_3 + p_5 - \ldots)^2$.

31. Prove that $x^3 + 3x - 3 = 0$ has one and only one real root α, and hence that $x^4 + 6x^2 - 12x - 9 = 0$ has just two real roots.

32. Find the range of k for which the following equations have 4 distinct roots. Illustrate the results by sketches.

(a) $x^4 - 14x^2 + 24x - k = 0$

(b) $3x^4 - 16x^3 + 6x^2 + 72x - k = 0$.

33. If $a_1, a_2, \ldots, a_n$ are all different, prove that the equation $\sum_{r=1}^{n} \frac{1}{x - a_r} = 0$ has exactly $(n - 1)$ roots

(a) by considering changes of sign of $p(x) = (x - a_1)(x - a_2)\ldots(x - a_n)\sum_{r=1}^{n} \frac{1}{x - a_r}$;

(b) by applying Rolle's theorem to $f(x) = (x - a_1) \ldots (x - a_n)$.

(You may assume, without loss of generality $a_1 < a_2 < \ldots < a_n$).

34. **(a)** If $p_n < 0$, and n is even, prove that the equation

$$p(x) \equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0$$

has at least one positive and at least one negative root.

(b) Prove that the number of positive roots of $p(x) = 0$ is odd if and only if $p_n < 0$.